December 2023 update

See below the latest updates for the current 20th edition, which are all incorporated in the PDF file of the Kourovka Notebook available at `https://kourovka-notebook.org` (see also `https://arxiv.org/pdf/1401.0300.pdf`).

For convenience of the readers all the updates since the first appearance of the 20th edition are also listed below; the newest comments or solutions added in December 2023 are marked **NEW**.

NEW ***8.75.** (A known problem). Suppose G is a finite primitive permutation group on Ω, and α, β are distinct points of Ω. Does there exist an element $g \in G$ such that $\alpha g = \beta$ and g fixes no point of Ω?

J. G. Thompson

*Not always (P. Müller, *Preprint*, 2023, `https://arxiv.org/pdf/2304.08459.pdf`).

12.48. Let G be a sharply doubly transitive permutation group on a set Ω (see Archive 11.52 for a definition).

(a) Does G possess a regular normal subgroup if a point stabilizer is locally finite?

(b) Does G possess a regular normal subgroup if a point stabilizer has an abelian subgroup of finite index?

Comments of 2022: an affirmative answer to part (b) is obtained in permutation characteristic 0 (F. O. Wagner, *Preprint*, 2022, `https://hal.archives-ouvertes.fr/hal-03590818`).

V. D. Mazurov

NEW ***14.4.** a) Is it true that there exists a nilpotent group G for which the lattice $\mathscr{L}(G)$ of all group topologies is not modular? (It is known that for abelian groups the lattice $\mathscr{L}(G)$ is modular and that there are groups for which this lattice is not modular: V. I. Arnautov, A. G. Topale, *Izv. Akad. Nauk Moldova Mat.*, **1997**, no. 1, 84–92 (Russian).)

b) Is it true that for every countable nilpotent non-abelian group G the lattice $\mathscr{L}(G)$ of all group topologies is not modular?

V. I. Arnautov

*a) Yes, it exists (V. Arnautov, A. Topală, *Bul. Acad. Ştiinţ. Repub. Moldova, Mat.*, 1998, no. 2(27), 130–131).

*b) No, there are such groups with modular $\mathscr{L}(G)$ (Dekui Peng, *Preprint*, 2023, `https://arxiv.org/abs/2310.08269`).

14.5. Let G be an infinite group admitting non-discrete Hausdorff group topologies, and $\mathscr{L}(G)$ the lattice of all group topologies on G.

NEW *d) Let G be an abelian group, k a natural number. Let A_k be the set of all those Hausdorff group topologies on G that, for every topology $\tau \in A_k$, any non-refinable chain of topologies starting from τ and terminating at the discrete topology has length k. Is it true that $A_k \bigcap \{\tau'_\gamma \mid \gamma \in \Gamma\} \neq \varnothing$ for any infinite non-refinable chain $\{\tau'_\gamma \mid \gamma \in \Gamma\}$ of Hausdorff topologies containing the discrete topology? (This is true for $k = 1$.)

V. I. Arnautov

*d) No; moreover, no infinite abelian group satisfies this property with $k = 2$ (Dekui Peng, *Preprint*, 2023, `https://arxiv.org/abs/2310.08269`).

14.42. Is a free pro-p-group representable by matrices over an associative-commutative profinite ring with 1?

A negative answer is equivalent to the fact that every linear pro-p-group satisfies a non-trivial pro-p-identity; this is known to be true in dimension 2 (for $p \neq 2$: A. N. Zubkov, *Siberian Math. J.*, **28**, no. 5 (1987), 742–747; for $p = 2$: D. E.-C. Ben-Ezra, E. Zelmanov, *Trans. Amer. Math. Soc.*, **374**, no. 6 (2021), 4093–4128).

A. N. Zubkov, V. N. Remeslennikov

14.53. *Conjecture:* Let G be a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure. Then G has an open subgroup H and an element t such that all elements of the coset tH have order dividing n.

This is true in the case $n = 2$. It would be interesting to see whether similar results hold for profinite groups in which the set of solutions of some equation has positive measure.

Comment of 2021: This is also proved for $n = 3$ (A. Abdollahi, M. S. Malekan, *Adv. Group Theory Appl.*, **13** (2022), 71–81).

L. Levai, L. Pyber

15.89. *Question made more precise:*
NEW Let Γ be an infinite undirected connected vertex-symmetric graph of finite valency without loops or multiple edges. Is it true that every complex number is an eigenvalue of the adjacency matrix of Γ under its natural action as a linear operator on the complex vector space of all complex-valued functions on the vertex set of Γ?

V. I. Trofimov

16.1. Let G be a finite non-abelian group, and $Z(G)$ its centre. One can associate a graph Γ_G with G as follows: take $G\backslash Z(G)$ as vertices of Γ_G and join two vertices x and y if $xy \neq yx$. Let H be a finite non-abelian group such that $\Gamma_G \cong \Gamma_H$.

b) If H is nilpotent, is it true that G is nilpotent?

Comment of 2009: This is true if $|H| = |G|$ (A. Abdollahi, S. Akbari, H. R. Maimani, *J. Algebra*, **298** (2006), 468–492).

NEW *Comment of 2023:* This is true if $|H| \geqslant |G|$ (H. Shahverdi, *to appear in J. Algebra*).

c) If H is solvable, is it true that G is solvable?

A. Abdollahi, S. Akbari, H. R. Maimani

17.37. Is there an integer n such that for all $m > n$ the alternating group A_m has no non-trivial A_m-permutable subgroups? (See the definition in 17.112.)

Comment of 2023: An affirmative answer is announced in (A. A. Galt, *Abstracts of Int. Conf. on Group Theory in honor of Victor Mazurov's 80-th birthday*, 2023, p. 38).

A. F. Vasil'ev, A. N. Skiba

***17.42.** Let $\overline{G}$ be a simple algebraic group of adjoint type over the algebraic closure $\overline{F}_p$ of a finite field F_p of prime order p, and σ a Frobenius map (that is, a surjective homomorphism such that $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ is finite). Then $G = O^{p'}(\overline{G}_\sigma)$ is a finite group of Lie type. For a maximal σ-stable torus $\overline{T}$ of $\overline{G}$, let $N = N_{\overline{G}}(\overline{T}) \cap G$. Assume also that G is simple and $G \not\cong \mathrm{SL}_3(2)$. Does there always exist $x \in G$ such that $N \cap N^x$ is a p-group?

E. P. Vdovin

*A definitive answer for a stronger question on the size of a base has been obtained in (T. C. Burness, A. R. Thomas, *J. Algebra*, **619** (2023), 459–504), which implies the following answer (in the notation therein): there is $x \in G$ such that $N \cap N^x$ is a p-group if and only if (G, N) is not one of the following: $(\mathrm{L}_3(2), 7{:}3)$, $(\mathrm{U}_4(2), 3^3{:}S_4)$, $(\mathrm{U}_5(2), 3^4{:}S_5)$.

18.89. Consider the set of balanced presentations $\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ with fixed n generators $x_1, \dots, x_n$. By definition, the group AC_n of Andrews–Curtis moves on this set of balanced presentations is generated by the Nielsen transformations together with conjugations of relators. Is AC_n finitely presented?

NEW *Comment of 2023*: It has been proved that AC_2 is not finitely presented (V. A. Roman'kov, *Preprint*, 2023, `https://arxiv.org/pdf/2305.11838.pdf`).

J. Swan, A. Lisitsa

***19.8.** A word in an alphabet $A = \{a_1, a_2, \dots, a_n\}$ is called a palindrome if it reads the same from left to right and from right to left. Let k a non-negative integer. A word in the alphabet A is called an *almost k-palindrome* if it can be transformed into a palindrome by changing $\leqslant k$ letters in it. (So an almost 0-palindrome is a palindrome.) Let elements of a free group $F_2 = \langle x, y \rangle$ be represented as words in the alphabet $\{x^{\pm 1}, y^{\pm 1}\}$. Do there exist positive integers m and c such that every element in F_2 is a product of $\leqslant c$ almost m-palindromes?

It is known that for $m = 0$ there is no such a number c. *V. G. Bardakov*

*No, there are no such integers (M. Staiger, *Preprint*, `https://arxiv.org/abs/2306.15752`).

***19.23.** For a group G, let $\mathrm{Tor}_1(G)$ be the normal closure of all torsion elements of G, and then by induction let $\mathrm{Tor}_{i+1}(G)$ be the inverse image of $\mathrm{Tor}_1(G/\mathrm{Tor}_i(G))$. The torsion length of G is defined to be either the least positive integer l such that $G/\mathrm{Tor}_l(G)$ is torsion-free, or ω if no such integer exists (since $G/\bigcup \mathrm{Tor}_i(G)$ is always torsion-free).

Does there exists a finitely generated, or even finitely presented, soluble group with torsion length greater than 2? *M. Chiodo, R. Vyas*

*Yes, such groups exist (I. J. Leary, A. Minasyan, *J. Group Theory*, **26**, no. 4 (2023), 741–750).

***19.24.** For a group G, let $\mathrm{Tor}(G)$ be the normal closure of all torsion elements of G. Does there exist a finitely presented group G such that $G/\mathrm{Tor}(G)$ is not finitely presented? Such a group must necessarily be non-hyperbolic. *M. Chiodo, R. Vyas*

*Yes, such groups exist: one soluble example and another virtually torsion-free are constructed in (I. J. Leary, A. Minasyan, *J. Group Theory*, **26**, no. 4 (2023), 741–750).

NEW ***19.38.** Suppose that H is a subgroup of a finite soluble group G that covers all Frattini chief factors of G and avoids all complemented chief factors of G. Is it true that there are elements $x, y \in G$ such that $H \cap H^x \cap H^y = H_G$, where H_G is the largest normal subgroup of G contained in H?

This is true if H is a prefrattini subgroup of G. *S. F. Kamornikov*

*Yes, it is true (S. F. Kamornikov, O. L. Shemetkova, *J. Algebra*, **641** (2024), 1–8).

19.53. *Question made more precise:*
NEW Let G be a group generated by elements x, y, z such that $x^3 = y^2 = z^2 = (xy)^3 = (yz)^3 = 1$ and $g^{12} = 1$ for all $g \in G$. Is it true that $|G| \leq 12$?

V. D. Mazurov

19.61. Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be an elementary carpet of type Φ over a commutative ring K (see 7.28), and let $\Phi(\mathfrak{A}) = \langle x_r(\mathfrak{A}_r) \mid r \in \Phi \rangle$ be its carpet subgroup. Define the *closure* of the carpet $\mathfrak{A}$ to be the set of additive subgroups $\overline{\mathfrak{A}} = \{\overline{\mathfrak{A}}_r \mid r \in \Phi\}$, where $\overline{\mathfrak{A}}_r = \{t \in K \mid x_r(t) \in \Phi(\mathfrak{A})\}$. Is the closure $\overline{\mathfrak{A}}$ of a carpet $\mathfrak{A}$ always a carpet?

An affirmative answer is known if $\Phi = A_l, D_l, E_l$. *Ya. N. Nuzhin*

***19.100.** Suppose that a finite group G admits a factorization $G = AB = BC = CA$, where A, B, C are abnormal supersoluble subgroups. Is G supersoluble?

A. F. Vasil'ev, T. I. Vasil'eva

*Yes, it is (L. S. Kazarin, V. N. Tyutyanov, *Prepint*, 2023).

***20.27.** Let G be a finite group, p a prime number, and let $|x^G|_p$ denote the maximum power of p that divides the class size of an element $x \in G$. Suppose that there exists a p-element $g \in G$ such that $|g^G|_p = \max_{x\in G} |x^G|_p$. Is it true that G has a normal p-complement?

A partial answer is in (`https://arxiv.org/abs/1812.03641`). *I. B. Gorshkov*

*No, not necessarily, a counterexample is given by `SmallGroup(192,945)` (B. Sambale, *Letter of 16 February 2022*).

20.58. Let $\omega(G)$ denote the set of element orders of a finite group G. A finite group G is said to be *recognizable* (*by spectrum*) if every finite group H with $\omega(H) = \omega(G)$ is isomorphic to G.

*a) Is it true that for every n there is a recognizable group that is the n-th direct power of a nonabelian simple group?

b) Is it true that there is a nonabelian simple group L such that for every n there is a recognizable group whose socle is the k-th direct power of L for some $k \geqslant n$?

V. D. Mazurov, A. V. Vasil'ev

*a) Yes, it is true (N. Yang, I. Gorshkov, A. Staroletov, A. V. Vasil'ev, *Annali Matem. Pura Appl.*, **202** (2023), 2699–2714).

NEW ***20.96.** Is a periodic group a Frobenius group if it has a proper non-trivial normal abelian subgroup that contains the centralizer of each of its non-identity elements?

A. I. Sozutov

*Yes, it is (D. V. Lytkina, V. D. Mazurov, *Siberian Math. J.*, **64**, no. 6 (2023), 1350–1353).

20.124. *A misprint in the definition corrected*:
A *Rota–Baxter operator* on a group G is a mapping $B : G \to G$ such that $B(g)B(h) = B\left(gB(g)hB(g)^{-1}\right)$ for all $g, h \in G$. Let F be a non-abelian free group. Is there a Rota–Baxter operator on F such that its image is equal to the derived subgroup $[F, F]$?

V. G. Bardakov

20.125. *The condition that G is non-abelian is added by the author*:
Does there exist a non-abelian group G and a Rota–Baxter operator $B : G \to G$ such that B is surjective but not injective?

V. G. Bardakov