Almost Engel conditions for finite, profinite and compact groups

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Engel groups

Notation: left-normed simple commutators

$$[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r].$$

Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x, g, g, \dots, g] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

Known facts on finite groups

Zorn's Theorem

A finite Engel group is nilpotent.

Baer's Theorem

If g is an Engel element of a finite group G,
that is, $[x, g, \dots, g] = 1$ for every $x \in G$, then $g \in F(G)$.

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.

Engel compact groups

J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zelmanov's Theorem

If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Almost Engel groups

Definition

A group G is almost Engel if for every $g \in G$ there is a finite set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \dots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

Includes Engel groups: when $\mathscr{E}(g) = \{1\}$ for all $g \in G$.

Theorem 1 (EKh and P. Shumyatsky)

Suppose that G is an almost Engel compact (Hausdorff) group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(...there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)

Three parts of the proof

1. **Finite groups**, a quantitative version.
2. **Profinite groups:** using finite groups, Wilson–Zelmanov theorem.
3. **Compact groups:** reduction to profinite case using structure theorems for compact groups.

Some notation

If G is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathscr{E}(g)$ with the property that for every $x \in G$,

$$[x, \underbrace{g, g, \dots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g)$$

(for possibly larger numbers $n(x, g)$).

We fix the symbols $\mathscr{E}(g)$ for these minimal sets, call them Engel sinks.

The nilpotent residual of a group G is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).

Almost Engel finite groups

Theorem 2 (EKh and P. Shumyatsky)

Suppose that G is a finite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$. Then $|\gamma_\infty(G)|$ is bounded in terms of m.

(G also has a nilpotent normal subgroup of bounded index: $C_G(\gamma_\infty(G))$.)

About the proof for finite groups

Lemma

In any almost Engel group G, the Engel sink is the set

$$\mathscr{E}(g) = \{z \in G \mid z = [z, g, \dots, g]\}$$

(with at least one occurrence of g).

Indeed, $x \to [x, g]$ is a mapping of $\mathscr{E}(g)$ into itself,

must be "onto" since $\mathscr{E}(g)$ is finite and minimal,

so any $z \in \mathscr{E}(g)$ belongs to its own orbit. □

About the proof for finite groups

Lemma

If $|\mathscr{E}(g)| \leqslant m$ for all $g \in G$, then $G/F(G)$ is of m-bounded exponent.

Proof: Clearly, g centralizes its powers. Hence for any $z \in \mathscr{E}(g^k)$ we have

$$z = [z, g^k, \dots, g^k] \quad \Rightarrow \quad z^g = [z^g, g^k, \dots, g^k].$$

Therefore $\mathscr{E}(g^k)$ is g-invariant.

Choose $k = m!$. Then $g^{m!}$ centralizes $\mathscr{E}(g^{m!})$,

hence $\mathscr{E}(g^{m!}) = \{1\}$ in fact, so $g^{m!}$ is an Engel element.

By Baer's theorem, then $g^{m!} \in F(G)$,

so $G/F(G)$ has exponent dividing $m!$. □

Further proof for finite groups

... Then it is shown that $|G/F(G)|$ is m-bounded.

Proof proceeds by induction on $|G/F(G)|$...

Recall:

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-p groups.

Largest normal pronilpotent subgroup (closed).

Lemma

A pronilpotent almost Engel group H is in fact an Engel group.

Proof: For any $h \in H$ there is a normal subgroup R

such that $\mathscr{E}(h) \cap R = \{1\}$ with nilpotent H/R.

Then $\mathscr{E}(h) \subseteq R$, so in fact $\mathscr{E}(h) = \{1\}$,

so h is an Engel element. □

Bounded version for profinite groups

Theorem 2 on finite groups immediately implies the following.

Corollary

Suppose that G is an almost Engel profinite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$.
Then G has a finite normal subgroup N
<u>of order bounded in terms of m</u> such that G/N is locally nilpotent.

General case of profinite groups

Theorem 3 (EKh and P. Shumyatsky)

Suppose that G is an almost Engel profinite group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Cannot simply apply Theorem 2 on finite groups – as there is no apriori uniform bound on $|\mathscr{E}(g)|$.

First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.

A piece of proof

Lemma

In an almost Engel profinite group G,

the sets $E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$ are closed.

Proof: For $y \notin E_k$ we have $|\mathscr{E}(y)| \geqslant k+1$,

so there are $z_1, z_2, \dots, z_{k+1}$ distinct elements in $\mathscr{E}(y)$, each

$$z_i = [z_i, y, \dots, y]. \qquad (*)$$

There is an open normal subgroup N such that the images of the z_i are distinct in the finite quotient G/N.

Then equations (*) show that for every $n \in N$ the sink $\mathscr{E}(yn)$ has an element in every coset $z_i N$, whence $|\mathscr{E}(yn)| \geqslant k+1$. So yN is also contained in $G \setminus E_k$. Thus, $G \setminus E_k$ is open, so E_k is closed. □

Application of Baire theorem

Recall: $E_k = \{x \mid |\mathscr{E}(x)| \leqslant k\}$ are closed.

In the theorem, G is almost Engel, which means $G = \bigcup E_k$.

By the Baire category theorem, one of E_k contains an open set, coset aU, where U is an open subgroup.

This gives us, in a certain metabelian section, a uniform bound for $|\mathscr{E}(u)|$ for all $u \in U$, and then Theorem 2 on finite groups can be applied...

It is then shown that $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above).

Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.

Compact groups

Recall

Theorem 1 (EKh and P. Shumyatsky)

Suppose that G is an almost Engel compact group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

Structure theorems for compact groups:

- The connected component G_0 of the identity is a divisible group (that is, for every $g \in G_0$ and every integer k there is $h \in G_0$ such that $h^k = g$).
- $G_0/Z(G_0)$ is a Cartesian product of simple compact Lie groups.
- G/G_0 is a profinite group.

Note that a simple compact Lie group is a linear group.

Scheme of proof for compact groups

An almost Engel divisible group is an Engel group.

By the structure theorem, G_0 is divisible, so is Engel by the above.

By well-known results (Garashchuk–Suprunenko, 1960), linear Engel groups are locally nilpotent.

Hence $Z(G_0) = G_0$ is abelian by the structure theorem.

Using the profinite case

Theorem 3 on profinite groups is applied to G/G_0.

Thus we have $G_0 < F < G$ with G_0 abelian divisible, F/G_0 finite, and G/F locally nilpotent.

Next steps:

$\mathscr{E}(g) \cap G_0 = \{1\}$ for all $g \in G$;

$[G_0, \mathscr{E}(g)] = 1$ for all $g \in G$;

Replace (rename) F by possibly smaller subgroup $\langle \mathscr{E}(g) \mid g \in G \rangle G_0$,

so $G_0 \leqslant Z(F)$;

... etc., in the end use Theorem 3 on profinite again.

Almost Engel in the sense of rank

Conjecture:

If G is a compact (or profinite) group in which $\mathscr{E}(g)$ generates a subgroup of finite (Prüfer) **rank** for every $g \in G$, then there is a normal closed subgroup N of finite rank such that G/N is locally nilpotent.

So far, the case of finite groups has been done:

Theorem 4 (EKh and P. Shumyatsky)

Suppose that G is a finite group and there is a positive integer r such that $\langle \mathscr{E}(g) \rangle$ has rank at most r for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of r.

An almost Engel element and some length parameters in finite groups

To measure 'deviation from being n-Engel':

Definition

$$E_n(g) = \langle [x, \underbrace{g, \dots, g}_{n}] \mid x \in G \rangle.$$

Remark: Note that this is not a subnormal subgroup, unlike the subgroups

$$G \trianglerighteq [G, g] \trianglerighteq [[G, g], g] \trianglerighteq \cdots$$

Soluble groups

Recall: Fitting series: $F_1(G) = F(G)$ largest normal nilpotent, then $F_{k+1}(G)/F_k(G) = F(G/F_k(G))$.

If G is finite soluble, then the least h such that $F_h(G) = G$ is the *Fitting height* of G.

Theorem 5 (EKh and P. Shumyatsky)

If g is an element of a soluble finite group G such that $E_n(g)$ (for some n) has Fitting height k, then $g \in F_{k+1}(G)$.

Generalized Fitting height

The generalized Fitting series of a finite group G starts from the generalized Fitting subgroup $F_1^*(G) = F^*(G)$, which the product of the Fitting subgroup and all quasisimple subnormal subgroups, and by induction $F_{i+1}^*(G)/F_i^*(G) = F^*(G/F_i^*(G))$.

The generalized Fitting height $h = h^*(G)$ of a finite group G is the least h such that $F_h^*(G) = G$.

Theorem 6 (EKh and P. Shumyatsky)

If g is an element of a finite group G such that $E_n(g)$ (for some n) has generalized Fitting height k, then $g \in F_{f(k,m)}^*(G)$, where m is the number of prime divisors of $|g|$.

Open problems

In the 'nonsoluble' Theorem 6, the function depends on the number of prime divisors of $|g|$. We conjecture that this dependence can be eliminated. Moreover, we have quite a precise conjecture (with best-possible bound):

Conjecture

Let g be an element of a finite group G, and n a positive integer. If the generalized Fitting height of $E_n(g)$ is equal to k, then $g \in F^*_{k+1}(G)$.

Reduction of conjecture

Question

Let $S = S_1 \times \cdots \times S_r$ be a direct product of nonabelian finite simple groups, and φ an automorphism of S transitively permuting the factors.
Is it true that $E_n(\varphi) = S$ for any n?

Theorem 7 (EKh and P. Shumyatsky)

Conjecture is true if the Question has an affirmative answer.

Some progress was made for the Question in the case where $|\varphi|$ is a prime by Robert Guralnick (unpublished).

Right-Engel analogues of length results

Let $R_n(g) = \langle [g, \underbrace{x, \dots, x}_{n}] \mid x \in G \rangle$.

Theorem 8 (EKh, P. Shumyatsky, and G. Traustason)

If G is a soluble finite group and the Fitting height of $R_n(g)$ is equal to k, then $g \in F_{k+1}(G)$.

Theorem 9 (EKh, P. Shumyatsky, and G. Traustason)

If the generalized Fitting height of $R_n(g)$ is equal to k, then $g \in F^*_{f(k,m)}(G)$, where m is the number of prime factors in $|g|$.

Compact groups all of whose elements are almost right-Engel

Definition

An element g of a group G is almost right-Engel if there is a finite set $\mathscr{R}(g)$ such that for every $x \in G$,

$$[g, \underbrace{x, \dots, x}_{n}] \in \mathscr{R}(g) \qquad \text{for all } n \geqslant n(x, g).$$

Theorem 10 (EKh and P. Shumyatsky)

If G is a compact (Hausdorff) group all of whose elements are almost right-Engel, then G has a finite normal subgroup N such that G/N is locally nilpotent.