Almost Engel compact groups

Evgeny Khukhro

Charlotte Scott Research Centre for Algebra
University of Lincoln, UK

Seminar talk at Royal Holloway
21 November 2018
Joint work with Pavel Shumyatsky
Engel groups

Notation: left-normed simple commutators

\[ [a_1, a_2, a_3, \ldots, a_r] = [[[a_1, a_2], a_3], \ldots, a_r]. \]

Recall: a group $G$ is an **Engel group** if for every $x, g \in G$,

\[ [x, g, g, \ldots, g] = 1, \]

where $g$ is repeated sufficiently many times depending on $x$ and $g$. Clearly, any locally nilpotent group is an Engel group.
Known facts on finite groups

**Zorn’s Theorem**

A finite Engel group is nilpotent.

**Proof:**

Coprime action $\Rightarrow$ non-Engel.

No coprime action $\Rightarrow$ nilpotent.

**Baer’s Theorem**

If $g$ is an Engel element of a finite group $G$, that is, $[x, g, \ldots, g] = 1$ for every $x \in G$, then $g \in F(G)$.

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.
J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zelmanov’s Theorem

If a Lie algebra $L$ satisfies a nontrivial identity and is generated by $d$ elements such that each commutator in these generators is ad-nilpotent, then $L$ is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.
**Definition**

A group $G$ is **almost Engel** if for every $g \in G$ there is a **finite** set $\mathcal{E}(g)$ such that for every $x \in G$,

$$[x, g, g, \ldots, g] \in \mathcal{E}(g)$$

for all $n \geq n(x, g)$.  

Includes Engel groups: when $\mathcal{E}(g) = \{1\}$ for all $g \in G$.

**Theorem 1 (almost Engel $\Rightarrow$ almost locally nilpotent)**

*Suppose that $G$ is an almost Engel compact (Hausdorff) group. Then $G$ has a finite normal subgroup $N$ such that $G/N$ is locally nilpotent.*

(...there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)
Three parts of the proof

1. **Finite groups**, a quantitative version.

2. **Profinite groups**: using finite groups, Wilson–Zelmanov theorem.

3. **Compact groups**: reduction to profinite case using structure theorems for compact groups.
If $G$ is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathcal{E}(g)$ with the property that for every $x \in G$,

$$[x, g, g, \ldots, g] \in \mathcal{E}(g) \quad \text{for all } n \geq n(x, g)$$

(for possibly larger numbers $n(x, g)$).

We fix the symbols $\mathcal{E}(g)$ for these minimal sets, call them Engel sinks.

The nilpotent residual of a group $G$ is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).
Almost Engel finite groups

For finite groups there must be a quantitative analogue of the hypothesis that the sinks \( \mathcal{E}(g) \) are finite.

**Theorem 2**

Suppose that \( G \) is a finite group and there is a positive integer \( m \) such that \( |\mathcal{E}(g)| \leq m \) for every \( g \in G \). Then \( |\gamma_\infty(G)| \) is bounded in terms of \( m \).

(\( G \) also has a nilpotent normal subgroup of bounded index: \( C_G(\gamma_\infty(G)) \).)

Evgeny Khukhro

Almost Engel compact groups
Lemma

In any almost Engel group $G$, the Engel sink is the set

$$\mathcal{E}(g) = \{ z \in G \mid z = [z, g, \ldots, g] \}$$

(with at least one occurrence of $g$).

Indeed, $x \rightarrow [x, g]$ is a mapping of $\mathcal{E}(g)$ into itself, must be “onto” since $\mathcal{E}(g)$ is finite and minimal, $z \in \mathcal{E}(g)$ belongs to its orbit.

Lemma

In a finite group, if $A$ is an abelian section, acted on by $g$ of coprime order, then $[A, g] = \{ [a, g, \ldots, g] \mid a \in A \}$ for any number of $g$, so $[A, g] \subseteq \mathcal{E}(g)$. 
Lemma

If $|\mathcal{E}(g)| \leq m$ for all $g \in G$, then $G/F(G)$ is of $m$-bounded exponent.

Proof: Clearly, $g$ centralizes its powers. Hence for any $z \in \mathcal{E}(g^k)$ we have

$$z = [z, g^k, \ldots, g^k] \Rightarrow z^g = [z^g, g^k, \ldots, g^k].$$

Therefore $\mathcal{E}(g^k)$ is $g$-invariant.

Choose $k = m!$. Then $g^{m!}$ centralizes $\mathcal{E}(g^{m!})$, hence $\mathcal{E}(g^{m!}) = \{1\}$ in fact, so $g^{m!}$ is an Engel element.

By Baer’s theorem, then $g^{m!} \in F(G)$, so $G/F(G)$ has exponent dividing $m!$. 

Evgeny Khukhro

Almost Engel compact groups
Further proof for finite groups

**Proposition**

If $\forall \, |\mathcal{E}(g)| \leq m$, then $|G/F(G)|$ is $m$-bounded.

First for the case of soluble $G$.

Then considering the generalized Fitting subgroup $= \text{socle of } G/S(G)$ (using CFSG)......

**Proof of Theorem 2** (that $|\gamma_\infty(G)|$ is $m$-bounded)

is by induction on $|G/F(G)|$...
Recall:

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-$p$ groups.

Largest normal pronilpotent subgroup (closed).
Lemma

A pronilpotent almost Engel group $H$ is in fact an Engel group.

Proof: For any $h \in H$ there is a normal subgroup $R$ such that $\mathcal{E}(h) \cap R = \{1\}$ with nilpotent $H/R$.

Then $\mathcal{E}(h) \subseteq R$, so in fact $\mathcal{E}(h) = \{1\}$, so $h$ is an Engel element.
Theorem 2 on finite groups immediately implies the following.

**Corollary**

Suppose that $G$ is an almost Engel profinite group and there is a positive integer $m$ such that $|E^k(g)| \leq m$ for every $g \in G$. Then $G$ has a finite normal subgroup $N$ of order bounded in terms of $m$ such that $G/N$ is locally nilpotent.
General case of profinite groups

**Theorem 3**

Suppose that \( G \) is an almost Engel profinite group. Then \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent.

Cannot simply apply Theorem 2 on finite groups – as there is no apriori uniform bound on \( |\mathcal{E}(g)| \).

First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.
Lemma

In an almost Engel profinite group $G$, the sets

$$E_k = \{ x \mid |\mathcal{E}(x)| \leq k \}$$

are closed.

Proof: For $y \notin E_k$ we have $|\mathcal{E}(y)| \geq k + 1$, so there are $z_1, z_2, \ldots, z_{k+1}$ distinct elements, each

$$z_i = [z_i, y, \ldots, y].$$

There is an open normal subgroup $N$ such that the images of the $z_i$ are distinct in the finite quotient $G/N$.

Then equations (*) show that for every $n \in N$ the sink $\mathcal{E}(yn)$ has an element in every coset $z_iN$, whence $|\mathcal{E}(yn)| \geq k + 1$. So $yN$ is also contained in $G \setminus E_k$. Thus, $G \setminus E_k$ is open, so $E_k$ is closed. \qed
Recall: $E_k = \{x \mid |\mathcal{E}(x)| \leq k\}$ are closed.

In the theorem, $G$ is almost Engel, which means $G = \bigcup E_k$.

By the Baire category theorem, one of $E_k$ contains an open set, coset $aU$, where $U$ is an open subgroup.

This gives us, in a certain metabelian section, a uniform bound for $|\mathcal{E}(u)|$ for all $u \in U$, and then Theorem 2 on finite groups can be applied...

Thus, $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above).

Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.
Recall

**Theorem 1**

Suppose that \( G \) is an almost Engel compact group. Then \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent.

**Structure theorems for compact groups:**

- The connected component \( G_0 \) of the identity is a divisible group (that is, for every \( g \in G_0 \) and every integer \( k \) there is \( h \in G_0 \) such that \( h^k = g \)).

- \( G_0/Z(G_0) \) is a Cartesian product of simple compact Lie groups.

- \( G/G_0 \) is a profinite group.

Note that a simple compact Lie group is a linear group.
Lemma

An almost Engel divisible group is an Engel group.

Proof: For \( g \in G_0 \), let \( |\mathcal{E}(g)| = m \). Choose \( h \in G_0 \) such that \( h^m = g \). Clearly, \( h \) centralizes \( g \), so for any \( z \in \mathcal{E}(g) \) we have

\[
z = [z, g, \ldots, g] \quad \Rightarrow \quad z^h = [z^h, g, \ldots, g].
\]

Hence \( \mathcal{E}(g) \) is \( h \)-invariant. Then \( h^m = g \) centralizes \( \mathcal{E}(g) \). This means that actually \( \mathcal{E}(g) = \{1\} \), so \( g \) is an Engel element.

\[\square\]

By the structure theorem, \( G_0 \) is divisible, so is Engel by the above.

By well-known results (Garashchuk–Suprunenko, 1960), linear Engel groups are locally nilpotent.

Hence \( Z(G_0) = G_0 \) is abelian by the structure theorem.
We apply Theorem 3 on profinite groups to $G/G_0$.

Thus we have $G_0 < F < G$ with $G_0$ abelian divisible, $F/G_0$ finite, and $G/F$ locally nilpotent.

Next steps:

$\mathcal{E}(g) \cap G_0 = \{1\}$ for all $g \in G$;

$[G_0, \mathcal{E}(g)] = 1$ for all $g \in G$;

Replace (rename) $F$ by possibly smaller subgroup $\langle \mathcal{E}(g) \mid g \in G \rangle G_0$, so $G_0 \leq Z(F)$;

... etc., in the end use Theorem 3 on profinite again.
Almost Engel in the sense of rank

Instead of being finite, suppose that $E(g)$ generates a subgroup of finite (Prüfer) rank, for all $g \in G$.

Conjecture:
If $G$ is a compact (or profinite) group, then there is a normal closed subgroup $N$ of finite rank such that $G/N$ is locally nilpotent.

So far, the case of finite groups has been done:

Theorem 4
Suppose that $G$ is a finite group and there is a positive integer $r$ such that $\langle E(g) \rangle$ has rank at most $r$ for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of $r$. 
Engel-type subgroups in finite groups and some length parameters

To measure ‘deviation from being $n$-Engel’:

**Definition**

$$E_n(g) = \langle [x, g, \ldots, g] \mid x \in G \rangle.$$  

**Remark:** Note that this is not a subnormal subgroup, unlike the subgroups

$$G \trianglelefteq [G, g] \trianglelefteq [[G, g], g] \trianglelefteq \cdots$$
Recall: Fitting series: $F_1(G) = F(G)$ largest normal nilpotent, then $F_{k+1}(G)/F_k(G) = F(G/F_k(G))$.

If $G$ is finite soluble, then the least $h$ such that $F_h(G) = G$ is the Fitting height of $G$.

**Theorem 5**

If $g$ is an element of a soluble finite group $G$ such that $E_n(g)$ (for some $n$) has Fitting height $k$, then $g \in F_{k+1}(G)$.

The proof of Theorem 1 reduces to the following proposition.

**Proposition**

Let $\alpha$ be an automorphism of a finite soluble group $G$ such that $G = [G, \alpha]$. Then $E_n(\alpha) = G$ for any $n$.

(Here, $E_n(\alpha)$ is a subgroup of $G\langle \alpha \rangle$.)
The generalized Fitting series of a finite group $G$ starts from the generalized Fitting subgroup $F_1^*(G) = F^*(G)$, which the product of the Fitting subgroup and all quasisimple subnormal subgroups, and by induction $F_{i+1}^*(G)/F_i^*(G) = F^*(G/F_i^*(G))$.

The generalized Fitting height $h = h^*(G)$ of a finite group $G$ is the least $h$ such that $F_h^*(G) = G$.

**Theorem 6**

If $g$ is an element of a finite group $G$ such that $E_n(g)$ (for some $n$) has generalized Fitting height $k$, then $g \in F_{f(k,m)}^*(G)$, where $m$ is the number of prime divisors of $|g|$.

(In fact, $f(k, m) = ((k + 1)m(m + 1) + 2)(k + 3)/2$.)
Non-soluble length

The nonsoluble length $\lambda(G)$ of a finite group $G$ is defined as the minimum number of nonsoluble factors in a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups.

Similarly to the generalized Fitting series, we can define terms of the ‘upper nonsoluble series’: $R_i(G)$ is the maximal normal subgroup of $G$ that has nonsoluble length $i$.

**Theorem 7**

*Let $m$ and $n$ be positive integers, and let $g$ be an element of a finite group $G$ whose order $|g|$ is equal to the product of $m$ primes counting multiplicities. If the nonsoluble length of $E_n(g)$ is equal to $k$, then $g$ belongs to $R_{g(k,m)}(G)$.  

(In fact, $g(k, m) = (k + 1)m(m + 1)/2$.)*
Importance of generalized Fitting height and nonsoluble length

Bounds for the nonsoluble length and/or generalized Fitting height greatly facilitate using the classification (and are themselves often obtained by using the classification).

Examples:

- reduction of the Restricted Burnside Problem to soluble and nilpotent groups in the Hall–Higman paper;

- Wilson’s reduction of the problem of local finiteness of periodic profinite groups to pro-$p$ groups;

(Both the Restricted Burnside Problem and the problem of local finiteness of periodic profinite groups were solved by Zelmanov.)

- our recent paper of EKh–Shumyatsky on similar problems about profinite groups.
Theorem 5 on generalized Fitting height follows from Theorem 6 on nonsoluble length and Theorem 4 on soluble groups.

The proof of Theorem 6 depends on the classification of finite simple group in so far as the validity of the Schreier conjecture on solubility of the group of outer automorphisms of a finite simple group.

One of the ingredients are properties of automorphisms of direct products of nonabelian finite simple groups. A typical lemma:

**Lemma**

Let $S = S_1 \times \cdots \times S_r$ be a direct product of $r$ isomorphic finite non-abelian simple groups and let $\varphi$ be the natural automorphism of $S$ of order $r$ that regularly permutes the $S_i$. Let $n$ be a positive integer. Then $E_n(\varphi) = S$. 
An important role in the proof is played by results on permutational actions of certain finite groups $G$ producing exact (regular) orbits of an element $g \in G$.

Corresponding lemmas rather too technical to be presented here...
Open problems and conjectures

In the ‘nonsoluble’ theorems the functions depend on the number of prime divisors of $|g|$. We conjecture that this dependence can be eliminated. Moreover, we have quite precise conjectures (with best-possible bounds):

Conjecture 1

Let $g$ be an element of a finite group $G$, and $n$ a positive integer. If the generalized Fitting height of $E_n(g)$ is equal to $k$, then $g \in F_{k+1}^*(G)$.

Conjecture 2

Let $g$ be an element of a finite group $G$, and $n$ a positive integer. If the nonsoluble length of $E_n(g)$ is equal to $k$, then $g \in R_k(G)$.
Reduction of conjectures

**Question**

Let $S = S_1 \times \cdots \times S_r$ be a direct product of nonabelian finite simple groups, and $\varphi$ an automorphism of $S$ transitively permuting the factors.

Is it true that $E_n(\varphi) = S$ for any $n$?

Thus, our **Lemma** above gives an affirmative answer in the special case where $|\varphi| = r$.

**Theorem 8**

Conjectures 1 and 2 are true if the Question has an affirmative answer.

Some progress was made for the Question in the case where $|\varphi|$ is a prime by Robert Guralnick (unpublished).